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ON EXPANSION OF THE POSSIBILITIES OF THE INTEGRAL TRANSFORMATION METHOD IN SOLVING PROBLEMS OF MECHANICS *

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The applicability of the integral transform method to a boundary value problem is related to whether the boundary conditions of this problem in the variable with respect to which the transformation is performed agree with the boundary conditions of the corresponding Sturm—Liouville problem. A method is proposed for transferring the integral transform method over to the case of boundary value problems when there is no such agreement. Moreover, the range of variation of the variable with respect to which the transformation is performed is smaller than the range of determination of the corresponding Sturm—Liouville problem, and the boundary condition of the boundary value problem is mixed. A method is indicated for transferring the mentioned method to boundary value problems in multiconnected and complex domains whose contours are inscribed in the coordinate mesh under consideration. The method is illustrated in anti-plane and plane problems of elasticity theory.

The aim and ideological side of the main content of the paper is convenient to be elucidated in application to boundary value problems for second order equations of general form (with separable variables) for the function $u \equiv u(x, y)$

$$r(x)(pu')' + r_1(y)(p_1u^{\cdot})^{\cdot} - q(x)u - q_1(y)u = 0 \quad (a_0 < x < a_1,\ b_0 < y < b_1) \tag{0.1}$$

Here and below the primes denote derivatives with respect to the first variable, and the dots with respect to the second variable.

It is known /1/ that to each integral transform which we write here in the general form

$$u_\lambda(y) = \int_{a_0}^{a_1} \frac{K(x, \lambda)}{r(x)} u(x, y)\,dx, \quad u = \int_l R(x, \lambda) u_\lambda(y)\,d\sigma \tag{0.2}$$

(l is the contour in the complex variable plane, $\sigma(z)$ is a distribution function /1/), there corresponds a Sturm—Liouville problem

$$\begin{gathered} r(pK')' - qK = -\lambda K \quad (a_0 < x < a_1) \\ U_j[k] = \alpha_{j_0} K(a_j, \lambda) + \alpha_{j_1} K'(a_j, \lambda) = 0 \quad (j = 0,1) \end{gathered} \tag{0.3}$$

with respect to the transformation kernel $K \equiv K(x, \lambda)$.

The known scheme of the integral transform method is based on agreement between the boundary conditions on the boundaries $x = a_j\ (j = 0,1)$ of the boundary value problem for (0.1) and the boundary conditions in (0.3). This permits /1/ reduction of the initial problem to a one-dimensional boundary value problem in the transformant $u_\lambda(y)$ by multiplying (0.1) by $r^{-1}(x) K(x, \lambda)$ and integrating by parts, if the boundary conditions on the boundaries $y = b_j\ (j = 0, 1)$ have the form

$$\begin{gathered} V_j[u] = \beta_{j_0} u(x, b_j) + \beta_{j_1} u^{\cdot}(x, b_j) = B_j(x) \\ (a_0 \leqslant x \leqslant a_1,\ j = 0,\ 1) \end{gathered} \tag{0.4}$$

i.e., are not mixed.

The next step /1/ in the scheme of the integral transform method (we call it classical) is to solve the one-dimensional boundary value problem obtained and to use the inversion formula from (0.2).

If the boundary conditions (0.4) on the boundaries $y = b_j$ become mixed, then the described scheme is dropped, there is no reduction to the one-dimensional boundary value problem, only the appropriate one-dimensional differential equation is used, its general solution is constructed, and a general integral or series representation of the solution of the initial boundary value problem is obtained on this basis. Then the mixed boundary conditions are realized, which result in dual integral or series equations. Such an approach to the solution of boundary value problems is used extensively at this time (see /2—4/, say).

There is a still greater deviation from the scheme of the integral transform method in situations when the boundary conditions on the boundary $x = a_j\ (j = 0,1)$ do not agree with the homogeneous Sturm—Liouville conditions (0.3), and even more so, if they turn out to be mixed. In such cases, general integral or series representations (/5/, say) which can be obtained either without integral transforms (the method of separation of variables) or /6/ by specially

*Prikl.Matem.Mekhan,44,No.1,130-142,1980 selected integral or series representations of solutions for simpler domains whose intersections will form that under consideration, are used to satisfy the boundary conditions.

There are also other (see /7.8/, for instance) approaches to the solution of mixed boundary value problems.

A means is indicated here (the first steps on this road were apparently taken in /9,10/) for realizing the classical scheme described above for the integral transform method in application to boundary value problems when the boundary conditions are mixed on the boundaries $y = b_j$ $(j = 0,1)$ while they do not agree with those described in (0.3) on the boundaries $x = a_j\ (j = 0, 1)$. Moreover, they may even be mixed and the range of variation of the appropriate variable in (0.1) can be less than the range in which the Sturm—Liouville problem is given. Besides, a method is indicated for transferring the classical scheme of the integral transform method to the case of multiconnected and complex domains.

The method elucidated here combined with the approach developed in /11/ broadens the range of applicability of the classical scheme of the integral transform method.

1. Let us consider a mixed boundary value problem for the equation (0.1) in the domain $a_0 \leqslant x \leqslant a \leqslant a_1,\ b_0 \leqslant y \leqslant b_1$, i.e., the domain of the change in the variable x is less than the domain (a_0, a_1) on which the Sturm—Liouville problem (0.3) is given.

In order not to complicate the situation, let us consider the boundary conditions on the boundaries $x = a_0$ and $y = b_0$ to be homogeneous and not mixed, i.e.,

$$U_0[u] = 0\ (b_0 \leqslant y \leqslant b_1),\ V_0[u] = 0\ \ (a_0 \leqslant x \leqslant a_1) \tag{1.1}$$

while we consider them mixed on the remaining boundaries. For instance, we give them in the following form on the boundary $x = a$

$$\begin{gathered} U^-[u] = \alpha_0^- u(a, y) + \alpha_1^- u'(a, y) = g_-(y)\ (b_0 \leqslant y < b) \\ U^+[u] = \alpha_0^+ u(a, y) + \alpha_1^+ u'(a, y) = g_+(y)\ (b < y \leqslant b_1) \end{gathered} \tag{1.2}$$

and on the boundary $y = b_1$ as

$$\begin{gathered} V^-[u] = \beta_0^- u(x, b_1) + \beta_1^- u^\cdot(x, b_1) = s_-(x)\ (a_0 \leqslant x < c) \\ V^+[u] = \beta_0^+ u(x, b_1) + \beta_1^+ u^\cdot(x, b_1) = s_+(x)\ (c < x \leqslant a) \end{gathered} \tag{1.3}$$

If the unknown functions $\chi_\mp(y),\ b_0 \leqslant y \leqslant b$ are introduced and we here consider

$$\chi_-, g_- \equiv 0\ \ (b < y \leqslant b_1),\ \chi_+, g_+ \equiv 0\ \ (b_0 \leqslant y < b)$$

then previous conditions can be written thus

$$\alpha_0^\mp u(a, y) + \alpha_1^\mp u'(a, y) = g_\mp + \chi_\pm\ \ (b_0 \leqslant y \leqslant b_1) \tag{1.4}$$

Analogously, by introducing the unknown functions $\psi_\mp(x),\ a_0 \leqslant x \leqslant a$ and considering

$$s_-, \psi_- \equiv 0\ (c < x \leqslant a),\ s_+, \psi_+ \equiv 0\ (a_0 \leqslant x < c)$$

we write in place of (1.3)

$$V^\mp[u] = s_\mp(x) + \psi_\pm(x)\ \ (a_0 \leqslant x \leqslant a) \tag{1.5}$$

The classical scheme of the integral transform method prescribes the reduction of the initial problem to a one-dimensional problem by using the integral transform (0.2). To perform this reduction, we consider $u(x, y) \equiv 0$ in the case being selected for $a < x \leqslant a_1$, and take the limit values from within the domain $a_0 < x < a_1,\ b_0 < y < b_1$ as the values of $u(x, y)$ and its derivatives at the point $x = a$.

Multiplying (0.1) by $r^{-1}(x) K(x, \lambda)$, we integrate by parts with respect to x in the segment $[a_0, a]$ (see /11/). The subsequent calculation of (0.2), (0.3) and (1.1) converts (0.1) to the following:

$$\begin{gathered} Lu_\lambda = r_1(p_1 u_\lambda')' - (q_1 + \lambda) u_\lambda = n_1(a) u(a, y) - n_0(a) u'(a, y) \\ (b_0 < y < b_1,\ \ n_1(a) = p(a) K'(a, \lambda),\ \ n_0(a) = p(a) K(a, \lambda)) \end{gathered} \tag{1.6}$$

We eliminate values of the desired function and its derivative at $x = a$ from the right side of the equation obtained. To do this we solve the system (1.4) for the values mentioned by considering $\Delta = \alpha_1^+\alpha_0^- - \alpha_0^+\alpha_1^- \neq 0$. It can be seen that the disturbance of this condition will result in a simpler situation when there is no point of boundary condition interchange on the boundary $x = a$. This particular case will also be considered below. Substituting the values $u(a, y)$ and $u'(a, y)$ found in this manner into (1.6), we will have

$$\begin{gathered} Lu_\lambda(y) = n^+[g_-(y) + \chi_+(y)] - n^-[\chi_-(y) + g_+(y)] \\ (b_0 < y < b_1,\ n^\pm = \Delta^{-1} p(a)\, U^\pm[K]) \end{gathered} \tag{1.7}$$

Application of the integral transform (0.2) to the boundary conditions (1.5) reduces them to the form

$$V^{\mp}[u_\lambda] = s_\lambda^{\mp} + \psi_\lambda^{\pm}, \quad \left\| \begin{matrix} s_\lambda^{\mp} \\ \psi_\lambda^{\mp} \end{matrix} \right\| = \int_{a_0}^{a} \frac{K(\xi, \lambda)}{r(\xi)} \left\| \begin{matrix} s_{\mp}(\xi) \\ \psi_{\mp}(\xi) \end{matrix} \right\| d\xi$$
$$V_0^{\cdot}[u_\lambda] = 0$$

Together with (1.7) these relationships show that the initial two-dimensional boundary value problem can be transformed to two versions of one-dimensional boundary value problems for the equation (1.7). In the first, the boundary conditions have the form

$$V_0[u_\lambda] = 0, \; V_1[u_\lambda] = V^+[u_\lambda] = s_\lambda^+ + \psi_\lambda^- \tag{1.8}$$

while in the second they should be taken in the form

$$V_0[u_\lambda] = 0, \; V_1[u_\lambda] = V^-[u_\lambda] = s_\lambda^- + \psi_\lambda^+$$

Which of these variants is taken depends on whether it is more convenient to take $\psi_-(x)$ or $\psi_+(x)$ as the desired function. For instance, if $\psi_-(x)$ is taken as the desired function, then the boundary value problem (1.7), (1.8) should be solved. Constructing the basis system /11/ of functions $\psi_0(y), \psi_1(y)$ and the Green's function $G(y, \eta)$ for it, we obtain the solution by means of the formula

$$u_\lambda(y) = \int_{b_0}^{b_1} G(y, \eta)\{n^+[g_-(\eta) + \chi_+(\eta)] - n^-[\chi_-(\eta) + g_+(\eta)]\}\, d\eta + \psi_1(y)(s_\lambda^+ + \psi_\lambda^-) \tag{1.9}$$

Using the inversion formula from (0.2), we express the solution of the mixed boundary value problem formulated in terms of the unknown functions $\chi_\pm(y)$, $\psi_-(x)$. Realization of the boundary conditions

$$V^-[u] = s_-(x) \;\; (a_0 \leqslant x < c)$$
$$U^-[u] = g_-(y) \;\; (b_0 \leqslant y < b), \quad U^+[u] = g_+(y) \;\; (b_0 < y < b_1) \tag{1.10}$$

results in a system of three integral equations in the functions mentioned.

Let us indicate a case when this quantity of equations is reduced. Let the boundary condition

$$U^-[u] = g(y) \;\; (b_0 \leqslant y \leqslant b_1) \tag{1.11}$$

hold in place of the mixed conditions (1.2). It can be seen that in this case we should set

$$g_- = g, \; \chi_- = u'(a, y) = \chi(y)$$
$$\chi_+ = g_+ = 0, \; \alpha_1^+ = 1, \; \alpha_0^+ = 0, \; \Delta = \alpha_0^-$$

in (1.9), and we obtain two equations to determine the desired functions χ, ψ_- by satisfying the boundary condition (1.11) as well as the first condition from (1.10).

Now, let $a = a_1$, i.e., the range of variation of the variable x in (0.1) and the range in which the Sturm—Liouville boundary value problem (0.3) is given, coincide. Meanwhile, if the mixed conditions still go over into just one $U_1[u] = g(y)$, $b_0 \leqslant y \leqslant b_1$, then we should set

$$n^+ = \alpha_{10}^{-1} p(a_1) K'(a_1, \lambda), \; g_- \equiv g, \; \chi_+ \equiv 0, \; n^- = 0$$

in (1.9), and only the function $\psi_-(x)$ remains unknown. We obtain its integral equation by satisfying the first condition from (1.10).

Remark 1. The parts of the series (improper integrals) not absolutely or weakly convergent should first be extracted and summed (evaluated) in realizing the conditions (1.10) and their particular cases by using (0.2) and (1.9).

Moreover, all the constructions performed evidently go over into the case when (0.1) is inhomogeneous or when the boundary condition on the boundary $x = a_0$ does not agree with the corresponding condition of the Sturm—Liouville problem, as well as the case when there are slits and point inclusions within the domain /11/.

2. We illustrate the above by the following antiplane problem of the theory of elasticity for a half-plane $(0 \leqslant x < \infty, -\infty < y < \infty)$ with a slit along the segment $y = 0, 0 \leqslant x \leqslant a$. Rigid stamps with flat bases are glued to symmetric segments $-b \leqslant y < 0, 0 < y \leqslant b$ of the elastic half-plane boundary $(x = 0)$. These stamps are subjected to longitudinal shear in opposite directions of the same quantity δ. It is required to find the stress distribution in the half-plane.

Since the line $y = 0$ is an axis of skew symmetry for the longitudinal displacements $u(x, y)$ of points of the half-plane, the problem posed can be formulated in the form of the following mixed boundary value problem for a quadrant:

$$u'' + u^{\cdot\cdot} = 0 \quad (0 < x < \infty, \; -\infty < y < \infty) \tag{2.1}$$
$$u(0, y) = -\delta \quad (0 \leqslant y \leqslant b), \;\; u'(0, y) = 0 \quad (b < y < \infty) \tag{2.2}$$
$$u^{\cdot}(x, 0) = 0 \quad (0 \leqslant x < a), \;\; u(x, 0) = 0 \quad (a \leqslant x < \infty) \tag{2.3}$$

It is here taken into account that the desired stresses are determined by the formulas (μ is the shear modulus)

$$\tau_{xz} = \mu u'(x, y), \quad \tau_{yz} = \mu u^{\cdot}(x, y) \tag{2.4}$$

If as above, the integral transform in the variable x is kept in mind in the case under consideration, the second boundary condition on the boundary $x = 0$ will correspond to the Fourier cosine transform

$$u_\lambda(y) = \int_0^\infty \cos\lambda x\, u(x, y)\, dx, \quad u = \frac{2}{\pi}\int_0^\infty u_\lambda(y)\cos\lambda x\, d\lambda \tag{2.5}$$

which we shall indeed use to reduce the formulated two-dimensional boundary value problem to a one-dimensional problem by predefining the boundary condition mentioned by using an unknown function (the contact stress under the stamp) $\chi_-(y) = \tau(y)$ which possesses the property $\chi_-(y) = \tau(y) \equiv 0$ $(b < y < \infty)$ on the whole boundary $x = 0$, i.e.

$$u'(0, y) = \mu^{-1}\tau(y) \quad (0 \leqslant y < \infty) \tag{2.6}$$

Subsequent multiplication of (2.1) by $\cos\lambda x$, integration by parts over the semi-axis $(0, \infty)$ and using (2.5) and (2.6) result in the equation

$$-u_\lambda'' + \lambda^2 u_\lambda = -\mu^{-1}\tau(y) \quad (0 < y < \infty) \tag{2.7}$$

As this follows from the general scheme elucidated above, there are two versions for ascribing boundary conditions to (2.7). In the case being examined these variants are determined by which of the conditions in (2.3) is predefined on the whole boundary. Let us predefine the second condition from (2.3)

$$u(x,0) = \varphi(x) \quad (0 \leqslant x < \infty) \tag{2.8}$$

by introducing the function $\psi_+(x) = \varphi(x)$, which is different from zero in a finite interval $(\psi_+ \equiv \varphi \equiv 0,\ x > a)$. Predefinition of the first condition from (2.3) in this case is less convenient since it results in a desired function different from zero in a semi-infinite interval.

Application of the transform (2.5) to (2.4) results in the following boundary value condition for (2.7):

$$u_\lambda(0) = \varphi_\lambda \quad \left(\varphi_\lambda = \int_0^\infty \cos\lambda\xi\, \varphi(\xi)\, d\xi\right) \tag{2.9}$$

Having used the funadmental function /11/

$$e_\lambda(y - \eta) = (2\lambda)^{-1} e^{-\lambda|y-\eta|} \tag{2.10}$$

to construct the Green's function

$$G(y, \eta) = e_\lambda(y - \eta) - e_\lambda(y + \eta) \tag{2.11}$$

of the boundary value problem (2.7), (2.9), we find its solution by means of the formula

$$u_\lambda(y) = -\frac{1}{\mu}\int_0^b G(y, \eta)\,\tau(\eta)\, d\eta + e^{-\lambda y}\varphi_\lambda \tag{2.12}$$

which is the analog of (1.9).

To execute further computations, it is convenient (compare with /11/) to transform (2.9) for φ_λ to the following by integration by parts:

$$\varphi_\lambda = -\int_0^a \frac{\sin\lambda\xi}{\lambda}\varphi'(\xi)\, d\xi$$

Keeping this as well as (2.10) and (2.11) in mind, we can write on the basis of (2.12)

$$u_\lambda'(y) = \frac{1}{\mu}\int_0^b \left[\frac{\operatorname{sgn}(y-\eta)}{e^{\lambda|y-\eta|}} - e^{-\lambda(y+\eta)}\right]\tau(\eta)\, d\eta + e^{-\lambda y}\int_0^a \sin\lambda\xi\varphi'(\xi)\, d\xi \tag{2.13}$$

Hence, by using the inversion formula from (2.5) and evaluating the known improper integrals (compare with /11/), we obtain

$$u^{\cdot}(x, y) = \frac{1}{\pi\mu}\int_0^b \left[\frac{y-\eta}{(y-\eta)^2 + x^2} - \frac{y+\eta}{(y+\eta)^2 + x^2}\right]\tau(\eta)\, d\eta + \frac{1}{\pi}\int_0^a \left[\frac{\xi - x}{(\xi - x)^2 + y^2} + \frac{\xi + x}{(\xi + x)^2 + y^2}\right]\varphi'(\xi)\, d\xi \tag{2.14}$$

A formula for the derivative $u'(x, y)$ in whose terms the stress τ_{xz} is expressed according to (2.4), can be obtained analogously.

To obtain the system of integral equations governing the functions $\tau(\eta)$ and $\varphi'(\xi)$, the first condition from (2.2) should be realized by writing it in the form $u^{\cdot}(0, y) = 0$ $(0 \leqslant y \leqslant b)$, and the first condition from (2.3). Having used (2.14) for this purpose, we obtain the required system of integral equations

$$-\frac{1}{\mu}\int_0^b \frac{\eta\tau(\eta)\,d\eta}{\eta^2 - y^2} + \int_0^a \frac{\xi\varphi'(\xi)\,d\xi}{\xi^2 + y^2} = 0 \quad (0 \leqslant y \leqslant b)$$

$$\frac{1}{\mu}\int_0^b \frac{\eta\tau(\eta)\,d\eta}{\eta^2 + x^2} - \int_0^a \frac{\xi\varphi'(\xi)\,d\xi}{\xi^2 - x^2} = 0 \quad (0 \leqslant x \leqslant a)$$

In the case $b = a$ this system is reduced to separately solvable (explicitly) singular integral equations by using addition and subtraction. The arbitrary constants in their solutions are found from the condition of integrability of the functions $\tau(\eta)$ and $\varphi'(\xi)$, as well as from the following easily verifiable condition

$$\int_0^a \varphi'(\xi)\,d\xi = \delta$$

Consequently, the solution of the system (2.15) for $b = a$ is written in the form

$$\frac{\tau(x)}{\mu} = \varphi'(x) = \frac{4\delta a\Gamma(^3/_4)}{\sqrt{\pi}\,\Gamma(^1/_4)\sqrt{a^4 - x^4}}$$

3. The constructions elucidated are even applicable in the case of boundary value problems for higher order equations then the second, and for systems, too. Let us illustrate this by an example of the first fundamental problem of plane elasticity theory for a rectangular domain $(-a \leqslant x \leqslant a, -\pi \leqslant y \leqslant \pi)$. The interval $(-\pi, \pi)$ is used to shorten the formulas. For this same reason we limit ourselves to the case when there is no load on the boundaries $y = \pm\pi$, while an identical tensile load $p(y) = p(-y)$ acts on the boundaries $x = \pm a$. The problem posed is equivalent to the following boundary value problem for the Airy function $u(x, y)$:

$$\Delta^2 u = 0 \quad (|x| < a,\ |y| < \pi) \tag{3.1}$$

$$u^{\cdot\cdot}(\pm a, y) = 0, \quad u^{\cdot\cdot}(\pm a, y) = p(y) \quad (|y| \leqslant \pi) \tag{3.2}$$

$$\sigma_y(x, \pm\pi) = u''(x, \pm\pi) = 0, \quad \tau_{yx}(x, \pm\pi) = -u^{\prime\cdot}(x, \pm\pi) = 0 \tag{3.3}$$

We note that the boundary conditions (3.3) are satisfied if

$$u(x, \pm\pi) = 0 \quad u^{\cdot}(x, \pm\pi) = 0 \tag{3.4}$$

Let us note that the function $p(y)$ is even by assumption, then $u(x, y)$ will also be an even function in the variable y, and we use the finite Fourier cosine transform in the variable y to solve the boundary value problem (3.1)—(3.4)

$$u_k(x) = \int_0^\pi u(x, y)\cos ky\,dy, \quad u(x, y) = \frac{u_0(x)}{\pi} + \frac{2}{\pi}\sum_{k=0}^{\infty} u_k(x)\cos ky \tag{3.5}$$

To do this we multiply the differential equation (3.1) by $\cos ky$ and integrate by parts. Taking account of the evenness of the function u in y and the boundary conditions (3.4), results in the equation

$$L_k u_k = u_k^{\mathrm{IV}} - 2k^2 u_k'' + k^4 u_k = f(x) \quad (|x| < a)$$
$$(f(x) = (-1)^{k+1}\chi(x), \quad \chi(x) = u^{\cdot\cdot\cdot}(x, \pm\pi)) \tag{3.6}$$

In the same manner the boundary conditions (3.2) are converted to the form

$$u_k(a) = U_0^*[u_k] = -k^2 p_k, \quad u_k(-a) = U_1^*[u_k] = -k^{-2} p_k \tag{3.7}$$

$$u_k'(a) = U_2^*[u_k] = 0, \quad u_k'(-a) = U_3^*[u_k] = 0$$

$$\left(p_k = \int_0^\pi \cos ky\, p(y)\,dy\right)$$

We construct the Green's function of the boundary value problem (3.6) and (3.7) by the scheme mentioned in /11/, and according to which it is determined by the formula

$$G_k(x, \xi) = \Phi_k(x, \xi) - \sum_{m=0}^{3} \psi_m U_m^*[\Phi_k], \quad \Phi_k(x, \xi) = \frac{1 + k|x - \xi|}{4k^3 e^{k|x-\xi|}}$$

where $\psi_m = \psi_m(x)$ $(m = 0, 1, 2, 3)$ is the basis system of functions

$$2\psi_0(x) = c_+(kx) + c_-(kx) + c_+(-kx) - c_-(-kx)$$
$$-2k\psi_2(x) = c_+'(kx) + c_-'(kx) + c_+'(-kx) - c_-'(-kx)$$
$$\psi_1(x) = \psi_0(-x), \quad \psi_3(x) = -\psi_2(-x)$$
$$c_\pm(x) = (\operatorname{sh} 2\alpha \pm 2\alpha)^{-1} [\operatorname{sh}(\alpha + x) + (\alpha - x)\operatorname{ch}(\alpha + x)],$$
$$\alpha = ak$$

which, as is easily verified, possesses the property

$$U_m^+[\psi_n] = \delta_{mn}, \quad L_k\psi_n = 0 \quad (m, n = 0, 1, 2, 3)$$

Using the Green's function and basis system constructed, the solution of the boundary value problem (3.6) and (3.7) can be written in the form /11,12/

$$u_k = (-1)^{k+1} \int_{-a}^{a} G_k(x, \xi)\chi(\xi)\,d\xi - \frac{p_k}{k^2}[\psi_0(x) + \psi_1(x)] \tag{3.8}$$

Using the inversion formula from (3.5) and summing the weakly converging series (compare with /11/), we find the function $u(x, y)$, and use it to find the stresses expressed in terms of the desired function $\chi(\xi)$. In particular, we will have $(\chi(\xi) = \chi(-\xi))$

$$\sigma_y = \frac{u_0''(x)}{\pi} + \int_{-a}^{a} \left[\frac{\partial S(x, \xi, y)}{\partial x} + R(x, \xi, y)\right]\chi(\xi)\,d\xi - Q'(x, y) \tag{3.9}$$

$$4\pi S(x, \xi, y) = (x - \xi)\{|x - \xi| - \ln[2\operatorname{ch}(x - \xi) + 2\cos y]\} + S_y(x, \xi) - S_y(-x, -\xi)$$

$$S_y(u, v) = (u - v)\ln[2\operatorname{ch}(2a - u - v) + 2\cos y] + 2(a - u)(a - v)\operatorname{sh}(2a - u - v)[\operatorname{ch}(2a - u - v) + \cos y]^{-1}$$

$$Q(x, y) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{c_+'(kx) - c_+'(-kx)}{k} p_k \cos ky$$

$$u_0''(x) = \int_{-a}^{a} \left(\frac{\xi^2 + a^2}{4a} - \frac{|x - \xi|}{2}\right)\chi(\xi)\,d\xi$$

$R(x, \xi, y)$ is a function which is differentiable as often as desired with respect to all the variables but for which we do not present any expression.

Having realized the first boundary condition from (3.3) by using (3.9), we obtain an integral equation to determine $\chi(\xi)$. After differentiating both sides, it becomes singular with the following structure

$$\int_{-a}^{a} \left[\operatorname{cth}\frac{x - \xi}{2} - H\left(\frac{a - x}{2}, \frac{a - \xi}{2}\right) - H\left(\frac{a + x}{2}, \frac{a + \xi}{2}\right) + R_1(x, \xi)\right]\chi(\xi)\,d\xi = -4\pi Q''(x, \pi)$$

$$(|x| \leqslant a,\ H(x, y) = 2h + (x - 5y)h' - 2xyh'',\ h = \operatorname{cth}(x + y))$$

As above, $R_1(x, \xi)$ here denotes a function which is arbitrarily differentiable with respect to both variables.

As we see, the kernel of the equation has a fixed singularity on the edges of the domain of definition, in addition to a movable singularity. Let us note that an integral equation of analogous structure has also been obtained in /6/.

4. Boundary value problems given in a rectangular domain (which can degenerate into a plane, a strip, a half-plane, etc.) were examined relative to the variables x, y which are not necessarily Cartesian. A method of transferring the classical scheme of the integral transform method to the case of more complicated domains, including multiconnected domains, is elucidated below. We again elucidate the idea in application to (0.1), but we pose the boundary value problem for a doubly-connected domain which is a rectangular domain $a_0 \leqslant x \leqslant a_1$, $b_0 \leqslant y \leqslant b_1$ with an extracted rectangle of smaller size $c_0 < x < c_1$, $h_0 < y < h_1$, i.e., $a_0 \leqslant c_0 < c_1 \leqslant a_1$, $b_0 \leqslant h_0 < h_1 \leqslant b_1$.

We take the boundary condition on the outer contour in the form

$$U_j[u] = 0 \quad (b_0 \leqslant y \leqslant b_1), \quad V_j[u] = 0 \quad (a_0 \leqslant x \leqslant a_1) \tag{4.1}$$

The boundary functionals U_j, V_j $(j = 0, 1)$ contained here are defined in (0.3) and (0.4). We consider the following boundary conditions given on the inner contour:

$$U_j^* [u] = \alpha_{j0}^* u (c_j, y) + \alpha_{j1}^* u' (c_j, y) = C_j (y) \tag{4.2}$$
$$(j = 0, 1;\ h_0 \leqslant y \leqslant h_1)$$
$$V_j^* [u] = \beta_{j0}^* u (x, h_j) + \beta_{j1}^* u^{\cdot} (x, h_j) = H_j (y) \tag{4.3}$$
$$(j = 0, 1;\ c_0 \leqslant x \leqslant c_1)$$

Considering (0.1) also given in the domain $c_0 < x < c_1,\ h_0 < y < h_1$, we introduce the notation for jumps in the function and its derivatives on the lines $x = c$ and $y = h$:

$$\langle u (c, y)\rangle = u (c - 0, y) - u (c + 0, y) \tag{4.4}$$
$$\langle u (x, h)\rangle = u (x, h - 0) - u (x, h + 0)$$

Then, assuming the function $u (x, y)$ continuous upon passing through the lines $x = c_j$ and $y = h_j$ $(j = 0, 1)$, and its normal derivative undergoes a break in continuity (another assumption is possible, continuity of the normal derivative and discontinuity of the function itself), we can write $(j = 0, 1)$

$$\langle u (c_j, y)\rangle = 0, \quad \langle u' (c_j, y)\rangle = \chi_j (y) \quad (b_0 \leqslant y \leqslant b_1) \tag{4.5}$$
$$\langle u (x, h_j)\rangle = 0, \quad \langle u^{\cdot} (x, h_j)\rangle = \varphi_j (x) \quad (a_0 \leqslant x \leqslant a_1) \tag{4.6}$$

where the unknown functions introduced should possess the property

$$\chi_j (y) \equiv 0, \quad y \overline{\in} (h_0, h_1), \qquad \varphi_j (x) \equiv 0, \quad x \overline{\in} (c_0, c_1)$$

We now apply the integral transform (0.2) to (0.1) and the second boundary condition from (4.1). As a result of executing the operations prescribed by the scheme in /11/ and taking account of (4.5), we obtain the following one-dimensional boundary value problem for the transformants:

$$Lu_\lambda (y) = -\sum_{j=0}^{1} n_0 (c_j) \chi_j (y) \quad (b_0 \leqslant y \leqslant b_1) \tag{4.7}$$
$$V_j [u_\lambda] = 0 \quad (j = 0, 1)$$

Application of the same transformation to the relations in (4.6) shows that the derivative of the solution for boundary value problem (4.7) at the points $y = h_j$ should undergo a break in continuity

$$\langle u_\lambda' (h_j)\rangle = \varphi_{j\lambda}, \quad \varphi_{j\lambda} = \int_{c_0}^{c_1} \frac{K (\xi, \lambda)}{r (\xi)} \varphi_j (\xi) \, d\xi \quad (j = 0,1) \tag{4.8}$$

The solution of the discontinuous boundary value problem (4.7), (4.8) will be constructed in the form

$$u_\lambda (y) = u_\lambda^0 (y) + v_\lambda (y) \quad (b_0 \leqslant y \leqslant b_1) \tag{4.9}$$

where $u_\lambda^0 (y)$ is the continuous part of the solution expressed in terms of the Green's function $G_* (y, \eta)$ of the self-adjoint boundary value problem

$$-(pu' (y))' + r_1^{-1} (y) [\lambda + q_1 (y)] u = f (y) \quad (b_0 < y < b_1) \tag{4.10}$$
$$V_j [u] = 0, \quad j = 0, 1$$

by means of the formula

$$u_\lambda^0 (y) = \sum_{j=0}^{1} \int_{h_0}^{h_1} \frac{G_* (y, \eta)}{r (\eta)} n_0 (c_j) \chi_j (\eta) \, d\eta \tag{4.11}$$

and $v_\lambda (y)$ is the discontinuous part of the solution of the boundary value problem (4.7), (4.8). To find it, we first solve the following discontinuous boundary value problem

$$-(pv')' + r^{-1} (\lambda + q_1) v = 0 \quad (b_0 < y < b_1), \ V_j [v] = 0 \tag{4.12}$$

with given jumps at the point $y = h$

$$\langle v' (h)\rangle = x_1, \quad \langle v (h)\rangle = x_0 \tag{4.13}$$

A method to solve similar problems is proposed in /11/. Using the scheme elucidated there, we arrive at the following formula to solve the discontinuous boundary value problem (4.12), (4.13) after a number of manipulations

$$v (y) = p (h) [x_1 G_* (y, h) - x_0 G_*^{\cdot} (y, h)] \tag{4.14}$$

The manipulations mentioned are not presented here because the validity of (4.14) is established easily by using the known properties /12/ of the Green's function for the self-adjoint boundary value problem (4.10). By using (4.14) and the discontinuous part $v_\lambda (y)$ of the boundary value problem (4.7), we can now write (4.8) in the form

$$v_\lambda (y) = \sum_{j=0}^{1} p_1 (h_j) G_* (y, h_j) \varphi_{j\lambda} \tag{4.15}$$

We obtain the solution of the initial boundary value problem expressed in terms of the unknown functions $\chi_j(y)$ and $\varphi_j(x)$ by means of the transform found (4.9), (4.11), (4.15) by using the inversion formula from (0.2). We obtain the system of integral equations for the determination by demanding compliance with the boundary conditions (4.2) and (4.3).

Remark 2. Here, as in Sect.1, Remark 1 remains valid. Moreover, the combination of what has been said here and in Sect.1 permits including the case of mixed boundary conditions on both the outer and inner contours. The elucidation above can be carried over to the case of higher order equations than (1.1), and also to systems.

In the case of the presence of a point ($x = c_0, y = h$) to replace the boundary conditions on the inner boundary $x = c_0, h_0 < y < h_1$, say, not one, as in (4.5) above, but two jumps should be introduced: for instance, a jump in the function itself before the point of boundary condition replacement, and in its derivative after the point.

5. Let us illustrate the use of the scheme in Sect.4 in such an antiplane problem for a half-plane $(-\infty < x < \infty, 0 \leqslant y < \infty)$ with a rectangular cutout $-c < x < c, 0 \leqslant y < h$ filled with an absolutely stiff medium (a deepened stamp), adhering completely (over the whole contour) to the elastic medium. The deepened stamp is subjected to the action of a given load shearing in the longitudinal direction. It is required to find the stress distribution in the elastic half-plane with the cutout described.

The problem posed is formulated mathematically in the form of equation (2.1) given in the quadrant $(x, y \geqslant 0)$ with a rectangular cutout $(0 \leqslant x < c,\ 0 \leqslant y < h)$ and the following boundary conditions

$$u'(0, y) = 0 \quad (h \leqslant y < \infty), \quad u^{\cdot}(x, 0) = 0 \quad (c \leqslant x < \infty) \tag{5.1}$$

$$u^{\cdot}(c, y) = 0 \quad (0 \leqslant y \leqslant h), \quad u'(x, h) = 0 \quad (0 \leqslant x \leqslant c) \tag{5.2}$$

In conformity with the scheme of Sect.4 let us consider (2.1) given in the whole quadrant with the exception of the lines $x = c$ and $y = h$ on which we give the jumps

$$\langle u(c, y)\rangle = 0, \quad \langle u'(c, y)\rangle = \chi(y) \quad (0 \leqslant y < \infty) \tag{5.3}$$

$$\langle u(x, h)\rangle = 0, \quad \langle u^{\cdot}(x, h)\rangle = \varphi(x) \quad (0 \leqslant x < \infty) \tag{5.4}$$

where

$$\chi(y) \equiv 0 \quad (h < y < \infty), \quad \varphi(x) \equiv 0 \quad (c \leqslant x < \infty)$$

Applying the integral transform (2.5) to (2.1), to the second condition from (5.1), as well as to (5.4), and taking (5.3) and the scheme of /11/ into account here, we arrive at a one-dimensional discontinuous boundary value problem

$$-u_\lambda'' + \lambda^2 u_\lambda = \cos\lambda c\ \chi(y) \quad (0 < y < \infty) \tag{5.5}$$

$$u_\lambda'(0) = 0, \quad \langle u_\lambda'(h)\rangle = \varphi_\lambda, \quad \left(\varphi_\lambda = \int_0^c \cos\lambda\xi\varphi(\xi)\, d\xi\right)$$

According to the scheme of Sect.4, to solve it, it is sufficient to construct the Green's function $G_*(y, \eta)$, decreasing at ∞, for the self-adjoint boundary value problem

$$-u''(y) + \lambda^2 u(y) = f(y) \quad (0 < y < \infty), \quad u'(0) = 0$$

It can be verified that this is the function

$$G_*(y, \eta) = e_\lambda(y - \eta) + e_\lambda(y + \eta) \tag{5.6}$$

where $e_\lambda(x)$ is defined by (2.10).

Using the Green's function constructed, we obtain the solution of the discontinuous boundary value problem (5.5) in the form

$$u_\lambda(y) = \cos\lambda c \int_{-h}^{h} e_\lambda(y - \eta)\chi(\eta)\, d\eta + [e_\lambda(y - h) + e_\lambda(y + h)]\varphi_\lambda \tag{5.7}$$

The function $\chi(y)$ is continued here in an even manner to negative values of the argument. Using (5.7) and the inversion formula from (2.5), we find the function $u(x, y)$ and its derivative

$$\left\| \begin{matrix} u^{\cdot}(x, y) \\ u'(x, y) \end{matrix} \right\| = \frac{2}{\pi} \int_0^\infty \left\| \begin{matrix} u_\lambda'(y)\cos\lambda x \\ -u_\lambda(y)\lambda\sin\lambda x \end{matrix} \right\| d\lambda \tag{5.8}$$

Substitution of the derivative of (5.7) under the integral sign in (5.8) and evaluation of the integral already encountered results in the formula

$$2\pi u^{\cdot}(x, y) = -\int_{-h}^{h} \left[\frac{(y-\eta)\chi(\eta)}{(y-\eta)^2 + (x-c)^2} + \frac{(y-\eta)\chi(\eta)}{(y-\eta)^2 + (x+c)^2}\right] d\eta -$$

$$\int_{-c}^{c} \left[\frac{(y-h)\varphi(\xi)}{(y-h)^2 + (x-\xi)^2} + \frac{(y+h)\varphi(\xi)}{(y+h)^2 + (x-\xi)^2}\right] d\xi$$

By using analogous operations we obtain the formula for $u'(x, y)$ also. Using it together with (5.9), we realize condition (5.1). Consequently, we arrive at a system of singular integral equations

$$\int_{-h}^{h}\left[\frac{1}{y-\eta}+\frac{y-\eta}{(y-\eta)^2+4c^2}\right]\chi(\eta)\,d\eta+\int_{-c}^{c}\Big\{\frac{y-h}{(y-h)^2+(x-c)^2}+$$
$$(y+h)\left[(y+h)^2+(x-c)^2\right]^{-1}\Big\}\varphi(\xi)\,d\xi=0\quad(|y|<b)$$
$$\int_{-h}^{h}\left[\frac{(x-c)\chi(\eta)}{(x-c)^2+(h-\eta)^2}+\frac{(x+c)\chi(\eta)}{(x+c)^2+(h-\eta)^2}\right]d\eta+\int_{-c}^{c}\Big\{\frac{1}{x-\xi}+$$
$$\left[(x-\xi)^2+4h^2\right]^{-1}(x-\xi)\Big\}\varphi(\xi)\,d\xi=0\quad(|x|<c)$$

In particular, by virtue of the symmetry of the problem we have $\varphi(x)\equiv\chi(x)$ for $c=h$, that results in one equation in $\chi(x)$ which acquires the following form after evident changes of variables

$$\int_{0}^{1}\left[\frac{1}{t-\tau}+\frac{t}{t^2+\tau^2}-\frac{1-t}{(1-t)^2+\tau^2}+\frac{t-\tau}{(t-\tau)^2+1}\right]\times$$
$$\chi\left[h(1-2\tau)\right]d\tau=0\quad(0<t<1)$$

The arbitrary constant which will be contained in the solution of this homogeneous singular integral equation is found from the equilibrium condition for the deepened stamp.

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